INTRODUCING MULTIPLE-PERIOD OPTIMIZATION

Breaking Through the Myopic Limitation of Traditional Mean-Variance Portfolio Optimization

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1 INTRODUCTION

The traditional portfolio optimization framework focuses on interactions between return, risk, transaction cost, and other terms or constraints *at a particular point in time.* It disregards any data outside the analysis period. For this reason, this traditional framework is sometimes regarded as "myopic" by critics.

In this paper, we introduce a new feature in the Barra Optimizer—the Multiple-Period Optimization (MPO), which explicitly incorporates and processes data for multiple periods. Numerous research papers on multiple-period optimization already exist, yet with varying focuses. Some formulate the multi-asset multi-period portfolio optimization problem as a stochastic control problem ([1]-[3]). Others employ dynamic programming to provide analytical solutions for the multi-period mean-variance optimization problem under specific settings ([4]-[5]). Many try to maximize a utility function that depends on the investor's final wealth ([2]-[6]). In contrast, the Barra Optimizer's approach to MPO is to maximize the sum of utilities of all periods, where the utility for each period is a typical mean-variance utility for that single period.

This paper is organized as follows. We begin by presenting the general framework of MPO in the Barra Optimizer, comparing and contrasting its features and limitations with the traditional single-period optimization (SPO). Then, we use several examples to demonstrate the various applications of MPO, emphasizing the different results obtained by using MPO and sequential SPO. We show that sequential SPO is inadequate and ad hoc at best in handling cross-period constraints. In our studies, MPO consistently yields the highest total utility in the presence of transaction costs or cross-period constraints by macro-balancing the return, risk, and transaction cost terms of all periods.

2 MULTIPLE-PERIOD OPTIMIZATION IS SINGLE OPTIMIZATION

Multiple-Period Optimization does not mean solving multiple optimization problems for different periods, either simultaneously or sequentially. On the contrary, in the Barra Optimizer the problem is viewed as a single optimization problem. However, it involves input data for multiple periods, and produces optimal holdings as well as trading lists for all periods.

2.1 THE FRAMEWORK

As Exhibit 1 shows, the objective function of the Multiple-Period Optimization is a weighted sum of the objective functions of its component periods. The objective function for each period is a standard mean-variance single-period objective function supported in the Barra Optimizer. Typically, it consists of a linear return term, a quadratic risk term, and a piece-wise linear transaction-cost term.[1]

The constraints of the Multiple-Period Optimization can be categorized into two groups—per-period constraints and cross-period constraints.

Per-period constraints are period-specific, and apply only to their respective periods, even though one or more periods may have the same or similar constraints. For example, cardinality and threshold constraints are per-period constraints, even though all periods may share the requirements that the maximum number of assets in the portfolio is 50 or the minimum holding level for any asset is 1 basis point. Most of the convex constraints supported in single-period optimization can be set as per-period constraints in MPO.

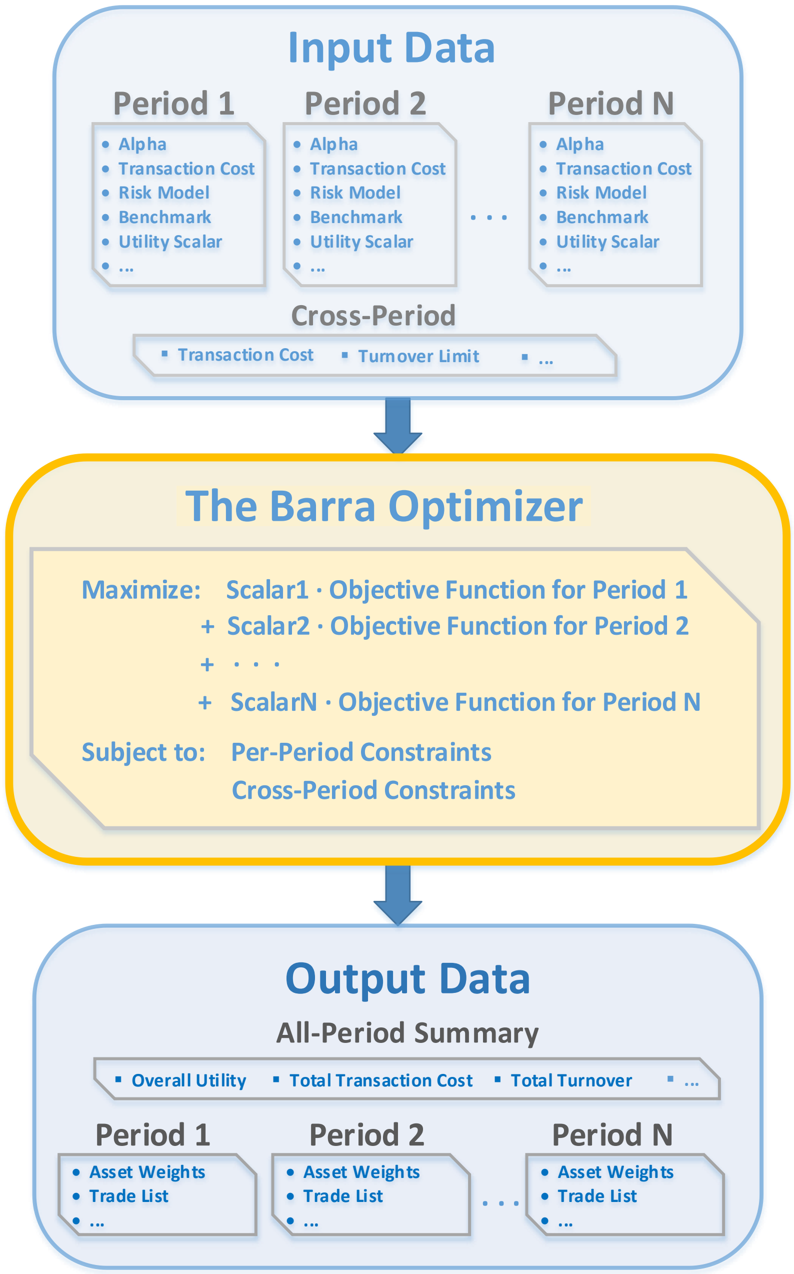
Cross-period constraints apply to at least two periods, and control certain interactions among these periods. An example of a cross-period constraint is the upper bound on the total transaction cost for all periods. Another cross-period constraint is the upper bound on the total turnover for the first two periods only.[2]

Multiple-Period Optimization requires users to provide input data for all periods upfront; it then returns the optimal portfolio and trade list for all periods all at once.

[1] For mathematical details of the single- and multiple-period optimization problems, please see the latest version of the Barra Optimizer User Guide.

[2] In the current release, however, cross-period constraints always apply to all periods.

Exhibit 1. The Multiple-Period Optimization Framework



2.2 COMPARING MPO WITH TRADITIONAL PORTFOLIO OPTIMIZATION

Multiple-Period Optimization shares many similarities with traditional portfolio optimization. However, there are also significant differences between the two types of optimization. Exhibit 2 summarizes the main similarities and differences.

Exhibit 2. Multiple-Period Optimization versus Traditional Portfolio Optimization

Characteristics	Multiple-Period Optimization	Traditional Portfolio Optimization
Decision Variables	The union of asset weights for all periods.[3]	Asset weights for a single-period portfolio only.
Objective Function	Weighted sum of the mean-variance objective functions for each period.	Mean-variance objective function for a single period.
Initial Portfolio	Pre-optimization (single-period) asset weights for the portfolio.	
Risk Model	Single-period asset or factor covariance matrices.	
Holding Constraint(s)	The sum of asset weights *for each period* equals 1. The number of holding constraints equals the number of periods.	The sum of asset weights equal to 1. There is one and only one holding constraint.
Linear and Factor Constraints	Can be set differently for each period, either at the portfolio or group level.	Can be set either at the portfolio or group level.
Cardinality and Threshold Constraints		
Turnover Constraints	Per-period as well as cross-period constraints are supported.	
Transaction Cost Constraints		

[3] Let $\mathbf{h}_p$ be the vector of asset weights and n_p be the number of assets in period p, respectively, $p = 1, 2, \cdots P$. Then, $\mathbf{h} = \begin{bmatrix} \mathbf{h}_1^T & \mathbf{h}_2^T & \cdots & \mathbf{h}_p^T \end{bmatrix}$ is the vector of decision variables for MPO, whose dimension is $n \times 1$, where $n = \sum_{p=1}^{P} n_p$. In other words, same assets in different periods are considered as different decision variables.

2.3 SCOPE AND LIMITATIONS

As indicated in Exhibit 1, the dimension of the Multiple-Period Optimization is much larger than the traditional portfolio optimization. Not only does the number of decision variables for asset holdings increase linearly with the number of periods, but new decision variables and constraints, as well as cross-period constraints, are also involved. These complexities make MPO computationally more challenging than traditional SPO. Exhibit 3 lists the scope of MPO in the current release.

Exhibit 3. Scope of Multiple-Period Optimization in the Current Release

Features	Comments
Number of Periods	A maximum of five.
Objective Function Terms	Convex quadratic only — no discrete or nonlinear terms.
Per-Period Constraints	Linear or convex piecewise linear only, with the exception of cardinality and threshold constraints.
Cardinality and Threshold Constraints	Holding cardinality and minimum holding threshold constraints supported for all periods, but trade cardinality and minimum trade threshold constraints supported only for the first period.
Cross-Period Constraints	Only two types supported—upper bounds on the total turnover and total transaction cost. Both apply to all periods.
Portfolio Base Value	For each period, portfolio base value must equal portfolio value, which equals the sum of beginning portfolio value plus cash flow value for that period. Users provide portfolio base value for the first period and cash flow weight for each period[4].
Risk Model	Only one risk model allowed in each period. The same risk model is used for all periods.
Benchmark	Only one benchmark allowed in each period. However, different periods may have different benchmarks.
Turnover Constraints	Overall as well as buy and sell-side turnover constraints are supported as per-period constraints. However, buy and sell-side cross-period turnover constraints are not supported.
Other Discrete, Nonlinear, or Nonconvex Constraints	Not supported, including but not limited to Risk Constraints, Leverage Constraints, and Roundlotting Constraints.

[4] Cash flow weight equals cash flow value divided by portfolio base value. Cash flow weight is a constant, whereas cash asset weight is a variable for each period.

3 MULTIPLE-PERIOD OPTIMIZATION HAS BROAD APPLICATIONS IN PORTFOLIO CONSTRUCTION

In the absence of transaction cost and cross-period constraints, MPO is equivalent to a sequence of SPOs with the initial portfolio of any future period being the optimal portfolio of the previous period. MPO is distinguished from SPO or a sequence of SPO's by its ability to define the interactions between periods within an integrated framework.

Put another way, input data for the current and future periods are considered by MPO as a whole, and information for future periods may influence the solution for the current period, if the periods are connected to each other through transaction costs or cross-period constraints. In this regard, MPO has "foresight" and the ability to macro-manage all periods, while SPO is simply "myopic".

In this section, we provide a few examples to demonstrate the broad applications of MPO in portfolio construction. We compare the results obtained using the MPO feature in Barra Optimizer and those obtained by solving the component single-period optimization problem sequentially from one period to another, and explain the differences. Case information is summarized in Exhibit 4.

Exhibit 4: Case Information

Base Information Applicable to All Cases	
• Universe: Cash + Randomly generated three assets from MSUS300 as of 2013/05/01 • Number of periods: 3 • Utility scalar for each period: 1 • Risk model: Barra USE4L • Initial portfolio: Asset weights are (0.0, 0.2, 0.4, 0.4) except in Case 3. The 1st asset is Cash. • Alpha: Randomly generated from the ranges [-0.01, 0.05], [-0.1, -0.01], [0.01, 0.1] for the three periods, respectively. The alphas in the four cases are all different. • Benchmark (when applicable): Benchmark assets are the same as those in universe, but with weights being (0.0, 0.4, 0.4, 0.2) • Transaction cost function (for all assets) when applicable: (h0 stands for the initial weight of an asset): Break Points: …, h0-0.4, h0-0.3, h0-0.2, h0-0.1, h0, h0_0.1, h0+0.2, h0+0.3, h0+0.4, … Slopes: …, -0.08, -0.06, -0.04, -0.02, 0.02, 0.04, 0.06, 0.08, … • Long-only portfolio (i.e., asset lower bounds are all zero) • Asset upper bounds are all 100% except the cash asset, which is capped at 20%. • Unless noted, no other linear or piecewise-linear constraints other than the holding constraint	
Case-Specific Information Differing from the Base	
Case 1	• Has transaction cost • No benchmark
Case 2	• No transaction cost • No benchmark • Has cross-period turnover constraint (total turnover <= 60%)
Case 3	• Has transaction cost • Has Benchmark • Initial portfolio is 100% cash • Has cross-period transaction cost constraint (total transaction cost <= 3.5%)
Case 4	• Has Benchmark • Cash for Period 2 is fixed at 30%, i.e., cash asset range for Period 2 is [0.3, 0.3]. Cash for other periods in MPO is unlimited, i.e., cash range is [0, 1].

3.1 CASE 1: INFLUENCE OF TRANSACTION COST

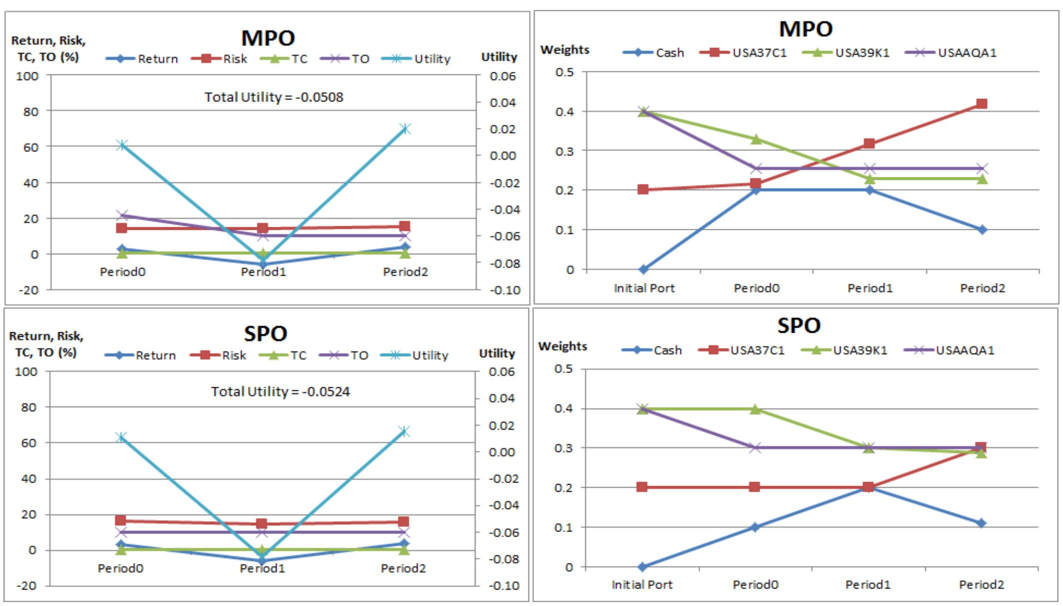
We first consider a case in which per-period transaction cost is included in the objective function, but there is no cross-period transaction cost or turnover constraints. This is a typical rebalance problem in which one starts with an existing portfolio and seeks the optimal portfolio weights for each period, given the information available for each period.

The top-left panel of Exhibit 5 shows the values of the objective terms—Return(%), Risk(%), Transaction Cost(%), Turnover (%) and Overall Utility—for each of the three periods in MPO. The bottom-left panel shows exactly the same values, but for each of the three periods in a sequential SPO. The sum of the overall utility is also noted as “total utility” in both panels.

The top-right panel of Exhibit 5 plots the optimal weight generated by MPO for each asset in each period. In comparison, the bottom-right panel plots the optimal weights produced by the sequential SPO.

As Exhibit 5 shows, the optimal weights prescribed by MPO are very different from those output by sequential SPO. Even though the return and risk values do not differ much between MPO and sequential SPO, the turnover values differ considerably, especially in Period 0. MPO’s optimal portfolios (for all three periods) have sacrificed the utility for Period 0 slightly, but in exchange produced a higher total utility for all periods. This is not surprising, since MPO’s objective is to maximize the total utility, and it does so by utilizing all information available to balance the risk, return, and transaction costs in all periods.

Exhibit 5: Case 1—Impact of Transaction Costs



3.2 CASE 2: TURNOVER ALLOCATION

In this example, we consider a case in which no transaction cost is included in the objective function, but there is a cross-period turnover constraint (total turnover <=60%). Again, this is a rebalance problem with an added requirement on total turnover.

Exhibit 6 compares the results of MPO with two versions of sequential SPO. In the “even” version of the sequential SPO, we impose 20% turnover limit each on Period 0, Period 1 and Period 2. In the “greedy” version, we first impose the full 60% turnover limit constraint on Period 0. However, since the resulting optimal solution for Period 0 only has a 40% turnover, 20% is left for use in Period 1 and Period 2. We again impose all 20% turnover on Period 1. This time, the turnover constraint is binding. In other words, the sum of the turnovers in Period 0 and Period 1 is already 60%, and we have to impose a zero turnover limit constraint on Period 2. This explains why the optimal solutions in Period 1 and Period 2 for the “greedy” SPO are identical, even though their return, risk, and utility values differ.

Exhibit 6: Case 2—Impact of Cross-Period Turnover Constraint

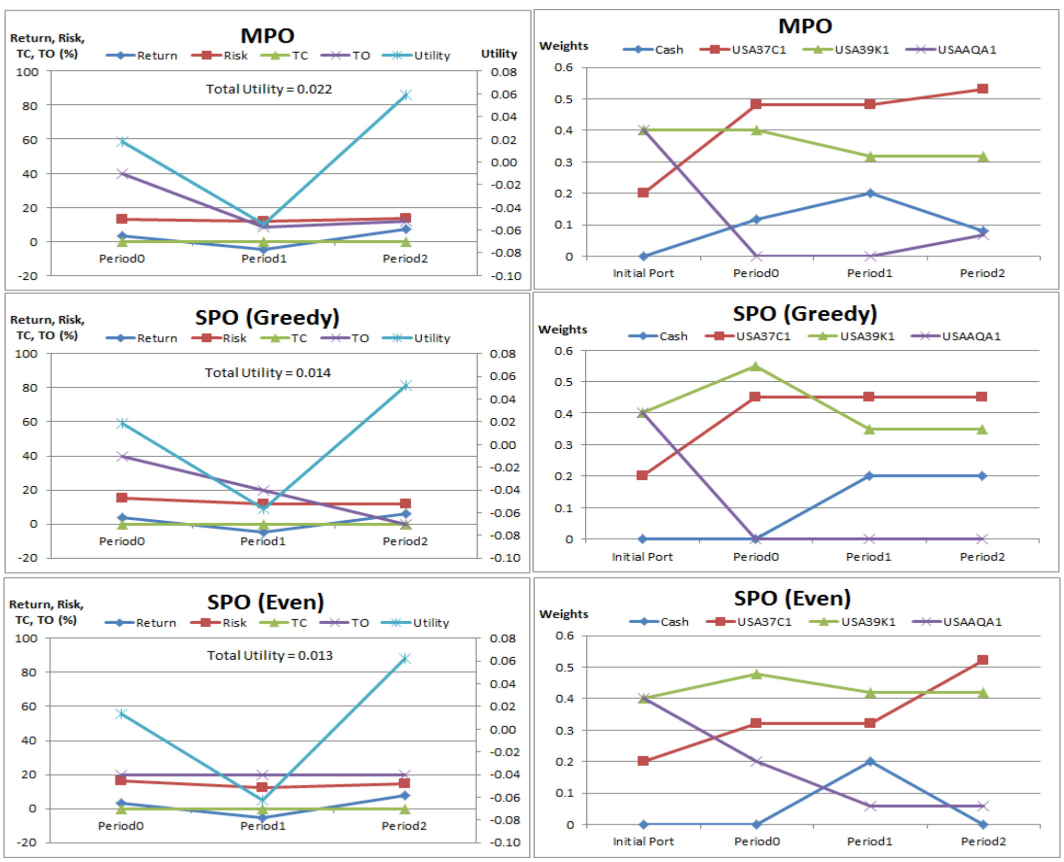


Exhibit 6 shows that MPO optimally allocates the amount of turnover among all periods to maximize the total utility. In this case, 40% is used in Period 0, 8% is used in Period 1, and 12% is used in Period 2. It does so systematically and automatically by taking into account data input for all periods. Sequential SPO, on the other hand, has no intrinsic way to handle cross-period constraints. The 'greedy' and 'even" allocations of total turnover are both arbitrary and naïve. It is not surprising that MPO yields the best total utility among the three approaches compared here.

3.3 CASE 3: TRADE SCHEDULING

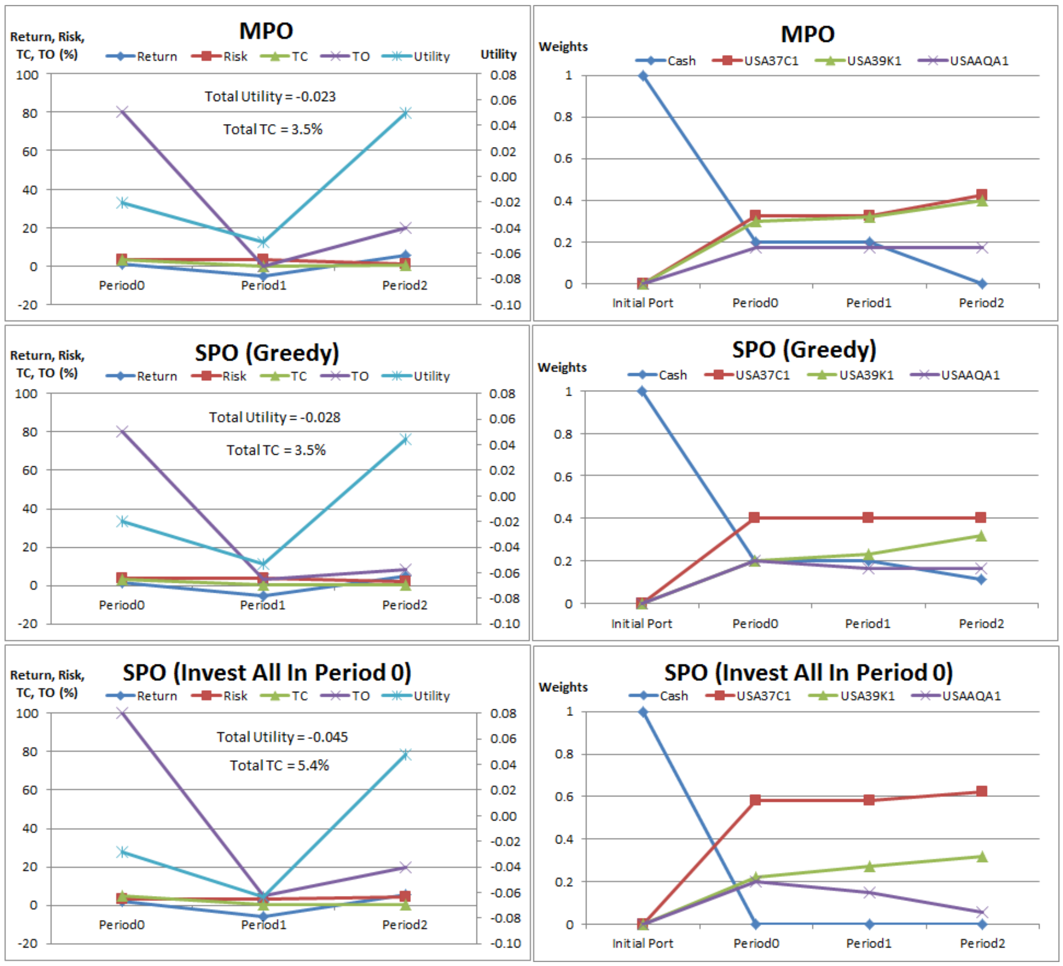
In this case, we start with an all-cash portfolio to track a benchmark. We want to know when and how much to invest in each period. Not only is per-period transaction cost included in the objective function, but there is also a cross-period transaction cost constraint (total transaction cost <=3.5%).

Again, since sequential SPO has no intrinsic way to handle cross-period constraints, we have tried three heuristics— "even", "greedy", and "invest all in Period 0". With the "even" heuristic, we impose one-third of the total transaction cost limit on each period. That is, each period has a transaction cost limit of 1.17%. Unfortunately, such a constraint immediately results in infeasibility for the first period in this case, due to limited turnover allowance corresponding to the transaction cost limit. There is no point in carrying out this heuristic further. With the 'greedy' heuristic, we first impose the total 3.5% transaction cost limit on Period 0, and then impose any unused transaction cost by the optimal solution of Period 0 on Period 1, and so on. With the "invest all in Period 0" heuristic, we change the cash asset range in Period 0 from [0, 0.2] to [0, 0], but do not impose any transaction cost constraint on any period.

Exhibit 7 compares the results of MPO with the "greedy" and "invest all in Period 0" two versions of sequential SPO. As expected, MPO has produced the best solution with the highest total utility, while satisfying the cross-period transaction cost constraint. It is interesting to note that even though the cash upper bound is set at 20% for MPO in all periods, the optimal portfolio produced by MPO for Period 2 is fully-invested. In contrast, about 12% cash remains in the optimal portfolio produced by the 'greedy' version of the sequential SPO for Period 2. In the "invest all in Period 0" version of the sequential SPO, optimal portfolios are fully invested for all three periods. However, the total transaction cost is 5.4%, exceeding the required 3.5%. In other words, the optimal solution produced by this heuristic violates the cross-period constraint, so it is essentially infeasible.

This example shows that ad hoc treatments of the cross-period constraints may easily lead to infeasibility. MPO is best suited for handling these constraints.

Exhibit 7: Case 3—Impact of Cross-Period Transaction Cost Constraint

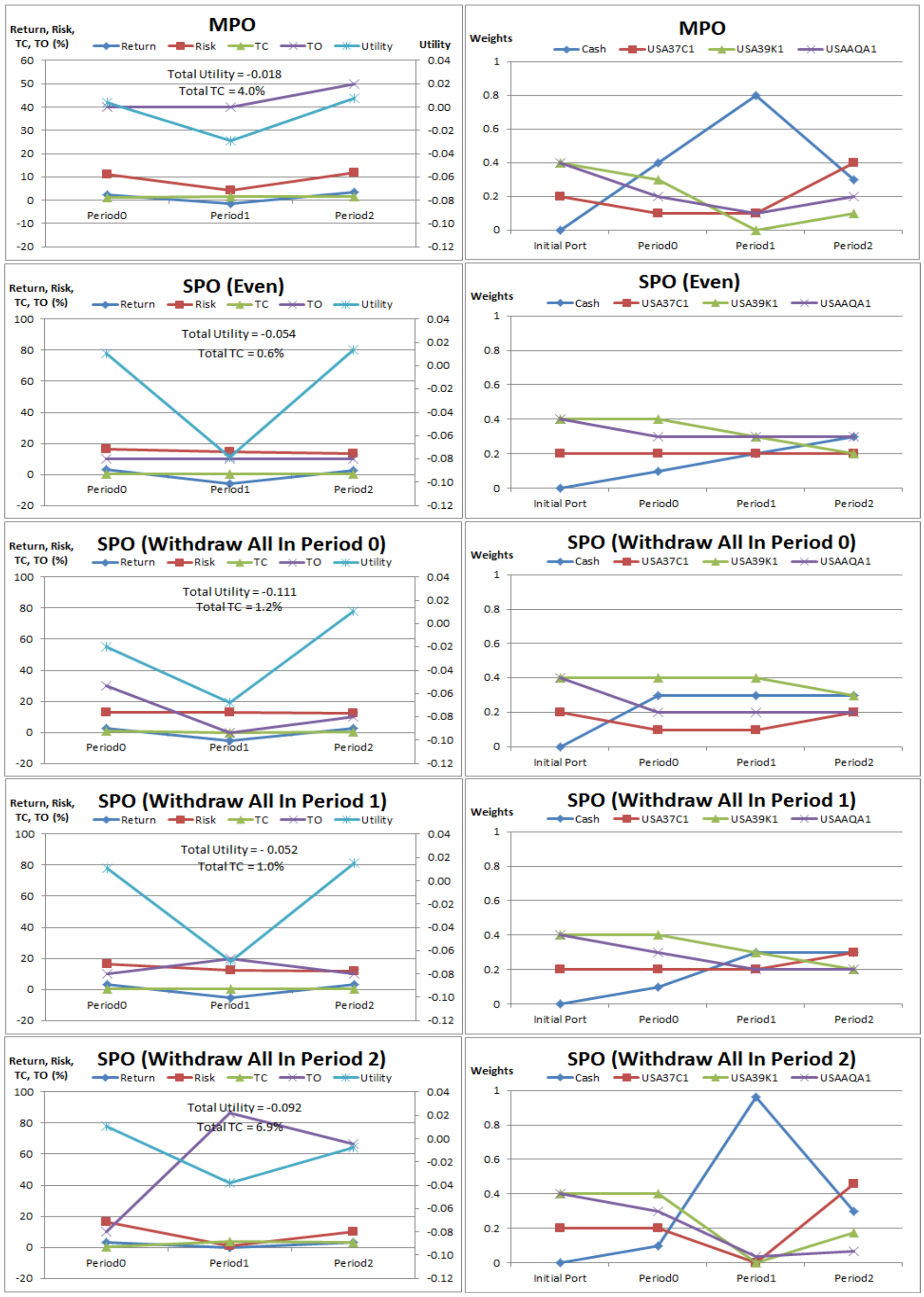


3.4 CASE 4: CASH FLOW MANAGEMENT

In this last case, we start with a fully-invested portfolio and we want to withdraw 30% cash at the end of Period 2. We want to know how much to withdraw in each period in order to maximize the total utility for all periods. Per-period transaction cost is included in the objective function, but there is no cross-period transaction cost or turnover constraint.

Exhibit 8 compares the results of MPO with several versions of naïve sequential SPO. In the "even" version, cash is withdrawn at the same rate—10% each period. In the other three

Exhibit 8: Case 4—Impact of Cash Bounds



versions, 30% cash is withdrawn all at once in one period only. It is interesting to see that the "even" version of the sequential SPO generated the smallest total transaction cost, but MPO still yields the highest total utility.

4 CONCLUSION

In the Barra Optimizer, Multiple-Period Optimization is treated as a single mean-variance optimization problem. Its objective is to maximize the weighted sum of the utilities of all periods, and decision variables are the joint portfolio holdings for all periods.

Due to cross-period constraints, as well as new decision variables and added constraints, Multiple-Period Optimization is typically much larger in size, and computationally more challenging, than the traditional single-period optimization, or even a sequence of single-period optimization problems. ***In the absence of transaction cost and any cross-period constraints, Multiple-Period Optimization may not be necessary*****,** since theoretically, a sequence of single-period optimization problems will yield the same results.

In the presence of transaction cost, however, Multiple-Period Optimization balances the risk, return, and transaction cost terms of all periods simultaneously to maximize the total utility of all periods. This is something a sequence of Single-Period Optimizations, each with a "myopic" nature, cannot do.

The benefit of Multiple-Period Optimization is especially pronounced when cross-period turnover or transaction cost constraints are applied. Single-Period Optimization has no systematic way of handling these constraints, and ad hoc or heuristic treatments may easily lead to infeasibility. Multiple-Period Optimization, on the other hand, "foresees" the necessary changes needed in all periods involved. From the perspective of MPO, these cross-period constraints are simply standard convex piece-wise linear constraints that can be handled easily and natively.

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